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Synchronization in Coupled Duffing Oscillators: Numerical Exploration using Exponential Time Differencing

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Abstract

The regularity that emerges from the interaction between irregular components is one of the fundamental questions in nonlinear physics. In this study, we examine how two externally driven, coupled Duffing oscillators can achieve synchronization. To analyze the dynamics of this system, we use the second-order Exponential Time Differencing (ETD2) numerical method, which allows us to comprehensively map the parameter space by varying the coupling strength (κ) and the driving force amplitude (γ). The results show that the synchronization process does not occur simply. The process only appears in a certain synchronization window at weak coupling ($\kappa \lesssim 0.2$). Within this region, the most prominent behavior is Phase Synchronized Chaos, where both oscillators achieve perfect phase locking (Synchronization Index ≈ 1), but their amplitudes remain chaotic and uncorrelated. We also find that the system is multistable, capable of exhibiting several different final states simultaneously. The boundaries between these basins of attraction are highly complex and exhibit fractal patterns, indicating extreme sensitivity to initial conditions. In other words, even if the system parameters remain constant, the final outcome remains difficult to predict. Overall, these findings provide a comprehensive picture of the dynamics of two coupled Duffing oscillators. The results highlight the balance between external driving forces, coupling strength, and intrinsic chaos that forms the basis for directed synchronization.

INTRODUCTION

The Duffing oscillator is one of the most fundamental paradigms in nonlinear dynamics, providing crucial insight into the behavior of systems where restoring forces deviate from simple harmonic motion. First introduced by Georg Duffing in 1918, this nonlinear oscillator has proven indispensable for understanding complex physical phenomena, ranging from structural vibrations in mechanical engineering to Josephson junction dynamics in superconductivity [1,2]. The richness of the system's dynamic behavior includes multiple period bifurcation, chaotic dynamics, and strange attractors, making it an essential model for studying nonlinear phenomena in various physical domains.

In physical systems, the Duffing equation naturally arises in contexts where nonlinear restoring forces dominate, such as stiffening springs in mechanical oscillators, large-amplitude vibrations in elastic structures, and particle dynamics in magnetic traps [3]. The forced, damped oscillator version exhibits highly complex behavior, reflecting real-world physical systems subjected to external driving forces, ranging from wind-driven structures to radio frequency-influenced circuits.

The model is very important from a theoretical point of view, because it provides an approximation of the dynamics observed in many real physical systems. For example, micro-electromechanical systems (MEMS) and nano-electromechanical systems (NEMS) exhibit nonlinear stiffness due to both geometric and material effects that leads to Duffing-type dynamics [4]. Similar nonlinear behaviors are observed in electrical circuits with nonlinear inductance or capacitance, as well as in large-amplitude vibrations of mechanical structures such as beams and plates [5].

Some of the applications of these models refer to phase-synchronized chaos (PSC) and to multistability. While they are usually regarded as mathematical curiosities, they have interesting physical implications. In particular, the observation of phase-synchronized chaos implies that the phase of the signals becomes coherent, while the phases remain stable. This phenomenon can be observed in coupled oscillatory devices. The same applies to multistability, which describes the phenomenon of the simultaneous existence of several possible states for some fixed parameter value. Multistability has been employed to provide switching mechanisms, although their unpredictability can lead to loss of control of the state of designing systems [6]. The analysis of the onset of these features is important to the design and control of nonlinear dynamical systems.

Synchronization of coupled nonlinear oscillators represents another fundamental phenomenon with profound physical implications. From synchronized firefly flashes to phase locking in array of lasers and the collective behavior of coupled pendulums, synchronization emerges as a universal principle in physical systems [7].

Comprehensive analysis of synchronization in coupled systems poses a unique challenge in theoretical physics. Early studies by Blekhman [8] discussed the possibility of chaos synchronization, while Boccaletti et al. [9] presented a comprehensive review of the phenomenon of synchronization in chaotic systems. Rosenblum et al. [10] made a significant contribution to phase synchronization analysis, developing a Hilbert transform-based technique that has become standard in physical applications.

Numerical studies of nonlinear oscillators have gradually evolved, with traditional methods such as Runge-Kutta and multistep approaches facing challenges in stiff systems and long-term integration [11]. The Exponential Time Differencing (ETD) method, pioneered by Certaine and significantly developed by Cox and Matthews [12], offers advantages for stiff systems by treating their linear terms exactly. ETD method, while effectively utilized across numerous domains in computational physics, has not been extensively employed for the analysis of coupled nonlinear oscillators. This is especially true when it comes to systematically investigating synchronization phenomena. Research conducted on coupled Duffing oscillators typically concentrates on particular synchronization regimes or limited parameter ranges. As a result, there is often a lack of comprehensive mapping of phase synchronization and multistability over a wide parameter space.

Several studies have examined synchronization in coupled Duffing oscillators, altough with some limitations. Davidsen et al. [13] investigated complete synchronization using active control methods, but focused on a limited parameter regime and used traditional numerical methods that can introduce artifacts in long-term simulations. Previous studies on nonlinear oscillators, including those summarized by Lakshmanan and Rajaseekar [14], have primarily focused on specific dynamical behaviors rather than systematic parameter-space mapping of synchronization phenomena. Similarly, the study by Ha et al. [15] on synchronization transitions only considered specific coupling configurations, while broader physical implications have not been adequately explored.

Over the past few years, significant progress have been made in grasping the concepts of synchronization and multistability within nonlinear dynamical systems. This is especially true when examining complex networks and chaotic oscillators. Recent research highlights the critical role of multistability and basin complexity in predicting the behavior of nonlinear systems [16]. Meanwhile, advancements in synchronization theory are continually enhancing our comprehension of stability and collective behaviors in coupled oscillators [17,18]. These findings highlight the necessity for thorough parameter-space exploration and the application of robust numerical methods to capture complex dynamical behaviors that might elude traditional methodologies.

Motivated by this gap, our current study sets out to explore how synchronization phenomena arise in coupled Duffing oscillators. We employ the second-order ETD method (ETD2) scheme, known for its numerical stability, to ensure accurate long-term integration. The study seeks to answer several key questions: (1) What conditions of coupling strength and driving amplitude lead to phase synchronization in chaotic Duffing systems? (2) Which form of synchronization predominates, and what is its connection to phase-synchronized chaos? (3) How does multistability affect the system's predictability and eventual state? Through addressing these questions, we aim to develop a more detailed understanding of how nonlinear dynamics, external forces, and coupling interact to influence synchronization behavior.

RESEARCH METHODS

Model: Coupled Duffing Oscillators

We investigate a system consisting of two identical, coupled, externally driven Duffing oscillators. This system is governed by the following set of coupled second-order ordinary differential equations (ODEs):

$$\frac{d^2x_1}{dt^2} + \delta\frac{dx_1}{dt} + \alpha x_1 + \beta x_1^3 = \gamma\cos(\omega t) + \kappa(x_2 - x_1) \tag{1}$$

$$\frac{d^2x_2}{dt^2} + \delta\frac{dx_2}{dt} + \alpha x_2 + \beta x_2^3 = \gamma\cos(\omega t) + \kappa(x_1 - x_2) \tag{2}$$

Where $x_1(t)$ and $x_2(t)$ are the positions of the two oscillators. The parameters are defined as:

- δ: Damping coefficient.
- α: Linear stiffness. We use $\alpha = -1.0$, which corresponds to a double well potential.
- β: Nonlinear stiffness, set to $\beta = 1.0$
- γ: Amplitude of the external periodic driving force.
- ω: Angular frequency of the driving force, set to $\omega = 1.0$
- κ: Coupling strength.

In this study, the parameters δ, α, β, and ω are kept constant, while the coupling strength κ and the driving force amplitude γ are the main control parameters to be explored.

Numerical Methods: ETD

To perform numerical integration, the second-order PDB system is first converted into a system of four first-order PDBs. We define the state vector as $u = [x_1, v_1, x_2, v_2]^T$, where $v_1 = dx_1/dt$ and $v_2 = dx_2/dt$. This allows the system to be written in a general semi-linear form as:

$$\frac{du}{dt} = Lu + N(u,t) \tag{3}$$

Here, L is a linear operator matrix, and N is a vector containing all nonlinear terms and driving force terms. Based on equations (1) and (2), their forms are:

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(\alpha+\kappa) & -\delta & \kappa & 0 \\ 0 & 0 & 0 & 1 \\ \kappa & 0 & -(\alpha+\kappa) & -\delta \end{pmatrix} \tag{4}$$

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$$N(u,t) = \begin{pmatrix} 0 \\ -\beta x_1^3 + \gamma\cos(\omega t) \\ 0 \\ -\beta x_2^3 + \gamma\cos(\omega t) \end{pmatrix} \tag{5}$$

We integrated this system using the second-order ETD method (ETD2), which is very stable and efficient for semi-linear problems [16]. The ETD2 recursion formula is given by:

$$u_{n+1} = \phi_0 u_n + h\phi_1 N_n + h\phi_2 (N_n - N_{n-1}) \tag{6}$$

Where $u_n = u(t_n)$, $N_n = N(u_n, t_n)$, and h is the time step dt. The function ϕ is derived from the matrix exponential e^{hL} [12]. As implemented in our code, this function ϕ is calculated using a Taylor series expansion to avoid potential singularities and numerical issues associated with matrix inversion when hL approaches zero. The first integration step is performed using the ETD1 scheme, since N_{n-1} is not yet available.

Synchronization Metrics

All simulations were run for a total time of T (for example, $T = 100$ until $T = 200$) with a time step of $dt = 0.01$. The first half of the resulting time series is discarded to ensure that the system has reached its final attractor and all transient behavior has decayed.

To quantify the relationship between oscillators, we measured two forms of synchronization:

- Complete Synchronization (CS): This is characterized by state convergence, namely $x_1(t) \rightarrow x_2(t)$ and $v_1(t) \rightarrow v_2(t)$. We track this using the synchronization error, $E(t) = |x_1(t) - x_2(t)|$. The final state where $\langle E(t) \rangle \approx 0$ indicates that CS has occurred.
- Phase Synchronization (PS): This is a weaker form in which the phase difference between oscillators becomes constant, $|\phi_1(t) - \phi_2(t)| \approx \text{constant}$, even though the amplitudes remain chaotic and uncorrelated. To measure this, we extract the instantaneous phase $\phi_i(t)$ from the time series of each oscillator $x_i(t)$.

From the phase difference $\Delta\phi(t) = \phi_1(t) - \phi_2(t)$, we calculate the synchronization index S to quantify the degree of phase locking.

$$S = \frac{1}{1 + \mathrm{Var}\big(\Delta\phi(t)\big)} \tag{7}$$

Where $\mathrm{Var}(\cdot)$ is the variance calculated during the post-transient time series. This index approaches $S = 1$ for perfect phase locking (zero variance) and $S = 0$ for unsynchronized and drifting phases.

RESULTS AND DISCUSSION

Through numerical investigation of coupled Duffing systems, we discovered diverse dynamics. The results show a prominent synchronization "window," clear dominance of chaotic phase synchronization, and strong evidence of multistability.

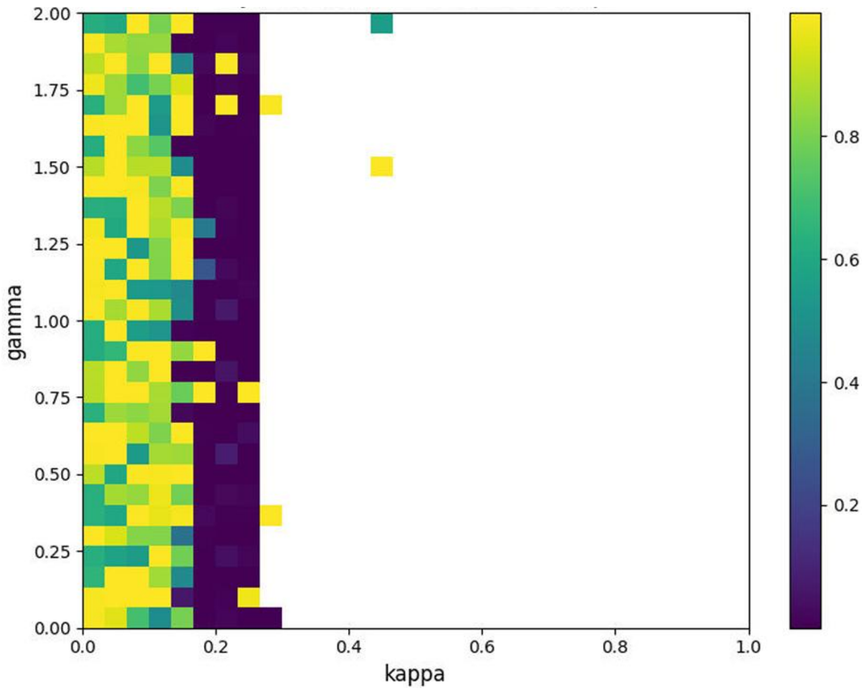

Figure 1. Synchronization index heatmap

The main findings of this study indicate that the synchronization level does not simply increase with coupling strength. Instead, synchronization only appears in certain “synchronization windows” in the parameter space. This phenomenon is most clearly seen in Figure 1, where areas with strong synchronization (green/yellow, with $S > 0.6$) appear prominent at relatively weak couplings, particularly in the region $\kappa < 0.2$. The appearance of a specific synchronization window at relatively weak coupling can be understood through the Master Stability Function (MSF) formalism, a concept by Louis Pecora and Thomas Carroll [20]. This framework examines how the stability of synchronized states is influenced by the interaction between the inherent dynamics of individual oscillators and the spectral characteristics of their coupling. Within chaotic systems, the MSF generally indicates that stable synchronization is achievable only within a certain range of coupling strengths, rather than increasing monotonically with coupling.

The non-monotonic synchronization pattern depicted in Figure 1, where synchronization is limited to a specific parameter range, aligns with these theoretical insights. Beyond this range, coupling that is too weak fails to counteract chaotic divergence, while overly strong coupling destabilizes the synchronized state. Although conducting a complete MSF calculation exceeds the scope of this study, the numerical findings we present strongly correspond with the established stability criteria from earlier research.

Interestingly, the standard deviation of the phase difference (Figure 2) shows that this region also has a high standard deviation, which is a strong indication of chaotic dynamics. The correspondence between these two results, namely the high synchronization index and the high chaos indicator, leads to the conclusion that this system is dominated by a phenomenon called Phase Synchronized Chaos (PSC).

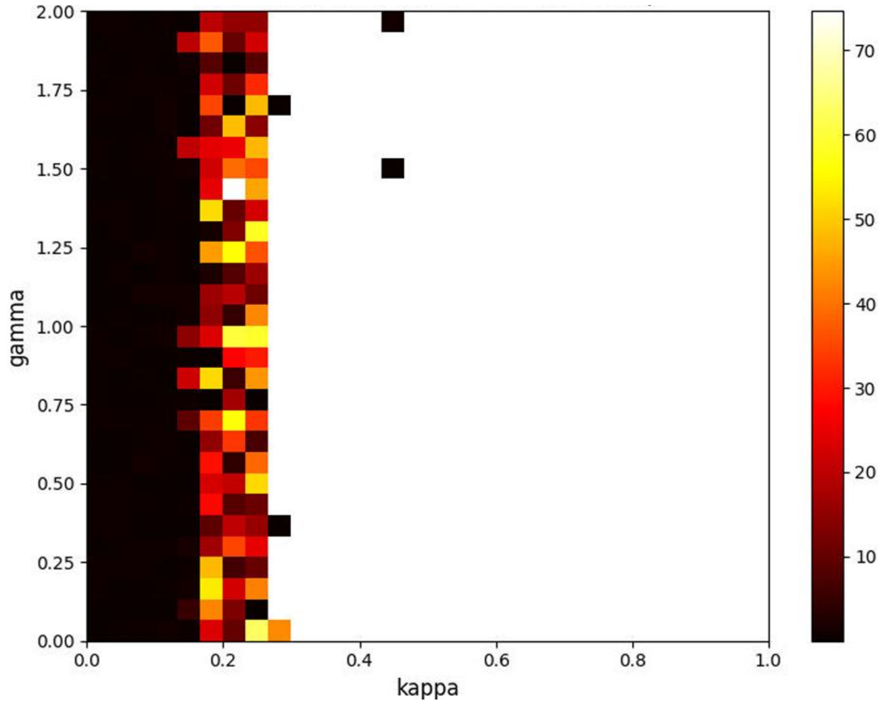

Figure 2. Standard deviation heatmap of phase difference

The simulation at $\kappa = 0{,}01$ (Figure 3) provides a very clear example of the PSC phenomenon. Under these conditions, the system achieves an almost perfect Phase Synchronization Index of 0.999, and is explicitly marked as "Phase Locked!". However, the final synchronization error remains quite large (≈ 0.64). This condition is characteristic of PSC, where both oscillators move in the same rhythm (locked in phase), but their amplitudes remain chaotic and uncorrelated.

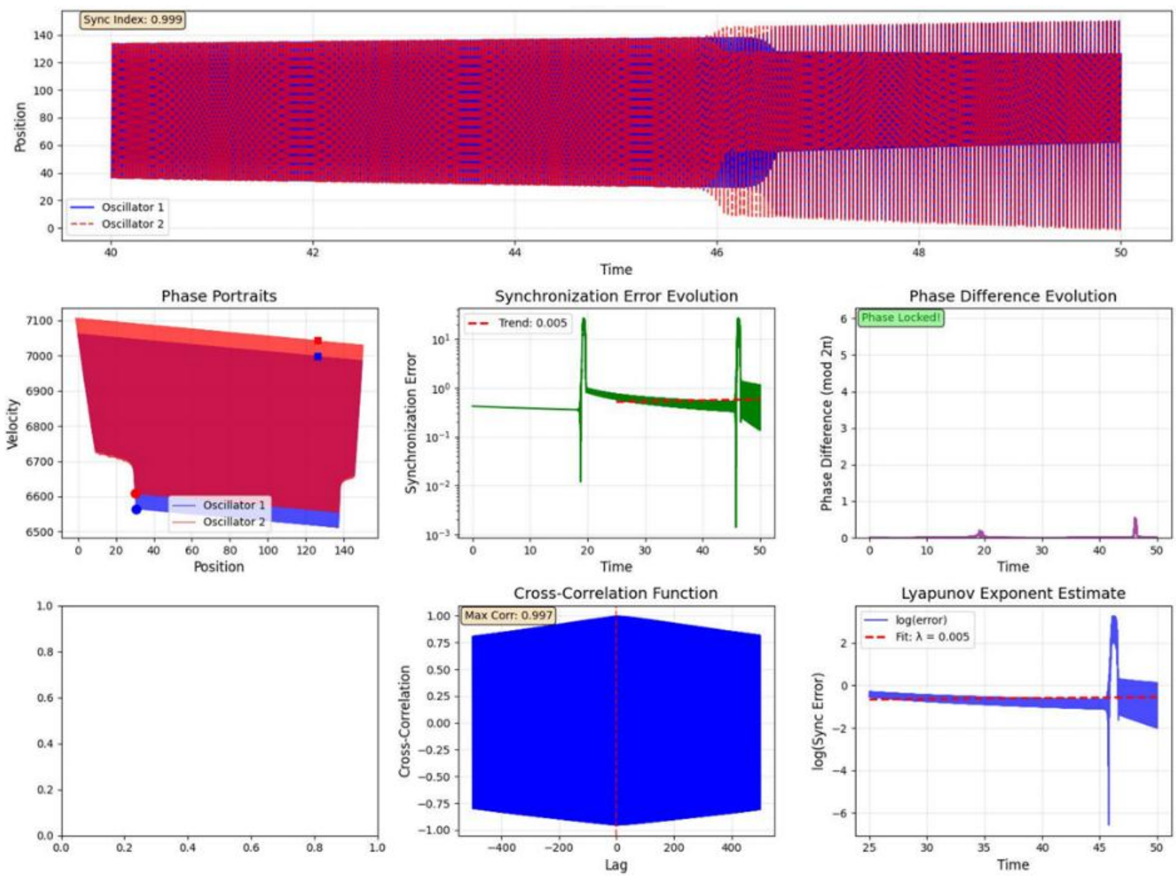

Figure 3. Results of the coupled Duffing oscillator analysis for $\kappa = 0.01$

Even within this seemingly stable synchronization "window," the final state of the system cannot be guaranteed. Figure 4, generated for $\kappa = 0.1$ where the value is within the PSC "window," shows evidence of multistability. The plot shows two large coexisting attractors: a red region with high S values leading to a synchronized state, and a dark blue "river" winding with low S values, leading to a desynchronized state.

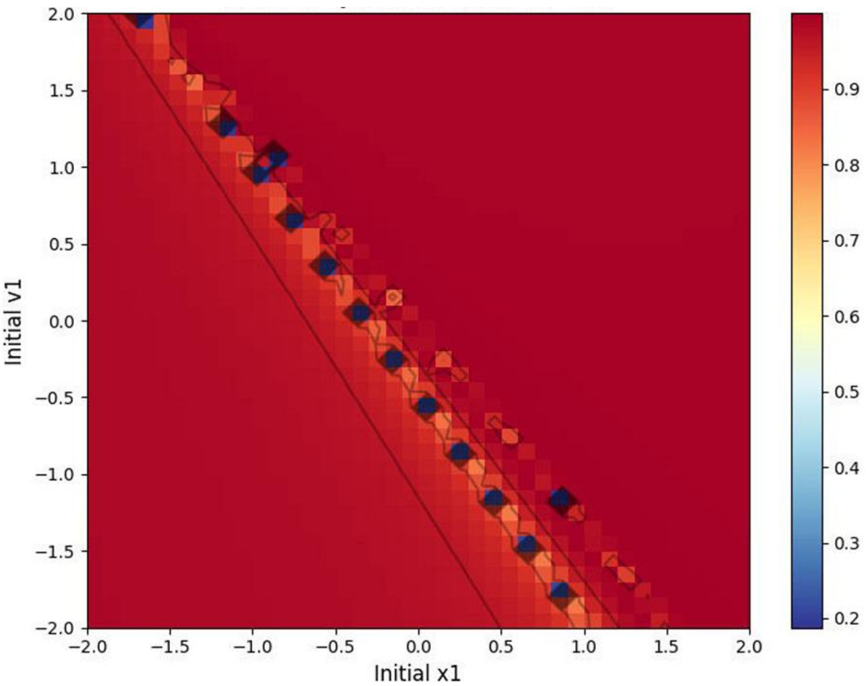

Figure 4. Basin of Synchronization plot for $\kappa = 0{,}1$

The boundary between these two regions appears complex and winding, as if it has a highly complex fractal structure. This configuration is very important, because it shows that even when the system parameters support synchronization, the final result is still highly dependent on the initial conditions. This extreme sensitivity shows that even the smallest disturbance near the boundary can drastically change the long-term behavior of the system, which is a characteristic of chaotic systems. In other words, we can never really predict whether synchronization will occur or not.

CONCLUSIONS

This study numerically investigates synchronization phenomena in coupled Duffing oscillators, focusing on the emergence of phase-synchronized chaos (PSC) and multi stability. The findings show that synchronization in chaotic systems occurs only within a limited parameter window, reflecting the complex interplay between inherent nonlinear instability and coupling-induced coherence. A key discovery is that the dominant collective behavior in this system is not complete synchronization, but rather Partial Synchronization Chaos (PSC), a state in which phase locking occurs alongside ongoing amplitude chaos. This confirms that synchronization in nonlinear systems is better understood as a spectrum of dynamic states rather than a simple binary transition. Furthermore, the presence of multi stability and fractal-like basin boundaries indicates that predictability remains inherently limited even when system parameters are fixed, due to extreme sensitivity to initial conditions. Methodologically, the use of the ETD2 scheme proves capable of providing stable, efficient long-term integration for exploring parameter space, making it essential for revealing complex dynamical structures such as synchronization windows.

Compared to previous studies, which have generally focused on complete synchronization or specific dynamical regimes, this work offers a broader perspective by systematically mapping phase synchronization and multi stability across parameter space, while extending understanding of the system's inherent predictability limits. Nevertheless, the study has several limitations, including its reliance on the ETD2 scheme, which may introduce numerical approximation errors in highly nonlinear dynamics, its restricted parameter range, and the basin sensitivity to initial conditions that has not been comprehensively sampled. Future research is therefore recommended to employ higher-order numerical schemes, broader parameter sweeps, and more systematic sampling strategies. These findings also have practical relevance for real-world systems such as MEMS resonators, nonlinear circuits, and mechanical structures, pointing toward future work on heterogeneous oscillator networks, time-delayed coupling, and stronger theoretical links such as stability function analysis. In the future, this approach can be extended to networks of non-identical oscillators, or to systems with time-delayed coupling effects.

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